**1038.Proposed by D. M. Bătinetu-Giurgiu, Matei Basarab National College, Bucharest,**
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Let $m$ be a nonnegative real number and $x, y$ be positive real numbers. Prove that, for any triangle $ABC$ with side lengths $a, b, c$ where $[ABC]$ denotes the area of triangle,

$$\frac{a^{m+2}}{(xb+yc)^m} + \frac{b^{m+2}}{(xc+ya)^m} + \frac{c^{m+2}}{(xa+yb)^m} \geq \frac{4\sqrt{3}}{(x+y)^m} \cdot [ABC].$$

**Solution by Arkady Alt , San Jose ,California, USA.**

Let $u := \dfrac{xb+yc}{x+y}, v := \dfrac{xc+ya}{x+y}, w := \dfrac{xa+yb}{x+y}$ and $I_m := \dfrac{a^{m+2}}{u^m} + \dfrac{b^{m+2}}{v^m} + \dfrac{c^{m+2}}{w^m}$

then $\sum_{cyc} \dfrac{a^{m+2}}{(xb+yc)^m} \geq \dfrac{4\sqrt{3}}{(x+y)^m} \cdot [ABC] \Leftrightarrow I_m \geq 4\sqrt{3} \cdot [ABC]..$

We will prove that $I_{m+1} \geq I_m$ for any $m \in \mathbb{N} \cup \{0\}$.

Noting that $I_0 = a^2+b^2+c^2$ and using inequality $\dfrac{\alpha^2}{\beta} \geq 2\alpha - \beta, \alpha, \beta > 0$ we obtain

$$I_1 = \sum_{cyc} \frac{a^3}{u} = \sum_{cyc} a \cdot \frac{a^2}{u} \geq \sum_{cyc} a(2a-u) = I_0 + \sum_{cyc} a(a-u) =$$

$$I_0 + \sum_{cyc}\left(a^2 - \frac{a(xb+yc)}{x+y}\right) = I_0 + a^2+b^2+c^2 - \sum_{cyc} \frac{a(xb+yc)}{x+y} =$$

$$I_0 + (a^2+b^2+c^2-ab-bc-ca) \geq I_0.$$

Taking inequality $I_1 \geq I_0$ as base of Math Induction and assuming for any $m \in \mathbb{N}$ that $I_m \geq I_{m-1}$ we will prove that $I_{m+1} \geq I_m$.

We have $I_{m+1} = \sum_{cyc} \dfrac{a^{m+3}}{u^{m+1}} = \sum_{cyc} \dfrac{a^{m+1}}{u^m} \cdot \dfrac{a^2}{u} \geq \sum_{cyc} \dfrac{a^{m+1}}{u^m}(2a-u) = I_m + \sum_{cyc}\left(\dfrac{a^{m+2}}{u^m} - \dfrac{a^{m+1}}{u^{m-1}}\right) =$

$I_m + (I_m - I_{m-1}) \geq I_m.$ ■

Since $(I_m)_{m\geq 0}$ is increasing sequence then $I_m \geq I_0 = a^2+b^2+c^2$ and

(**1**) $\quad a^2+b^2+c^2 \geq 4\sqrt{3} \cdot [ABC]$ (Weitzenböck's inequality)

we obtain $I_m \geq 4\sqrt{3} \cdot [ABC]$.

(Or, direct proof of inequality $a^2+b^2+c^2 \geq 4\sqrt{3} \cdot [ABC]$ :

Let $x := s-a, y := s-b, z := s-c$ where $s$ is semiperimeter and let $p := xy+yz+zx$, $q := xyz$. Also, assume (due to homogeneity) that $s := 1$. Then $a^2+b^2+c^2 = 2(1-p)$, $[ABC] = \sqrt{q}$ and inequality (**1**) become $1-p \geq 2\sqrt{3} \cdot \sqrt{q}$.

Since $p^2 = (xy+yz+zx)^2 \geq 3xyz(x+y+z) = 3q$ and

$1 = (x+y+z)^2 \geq 3(xy+yz+zx) = p$

we obtain $1-p-2\sqrt{3} \cdot \sqrt{q} = 1-3p+2\left(p-\sqrt{3q}\right) \geq 0$).